

Generalization of the Van Cittert–Zernike theorem: observers moving with respect to sources

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Abstract. The use of the Van Cittert–Zernike theorem for the formulation of the visibility function in satellite-based Earth observation with passive radiometers does not take into account the relative motion of the observer (the satellite antenna) with respect to sources of the electro-magnetic fields at the surface of the Earth. The motion of the observer leads on the one hand to a more complex signal due to a pixel-dependent Doppler shift that is neglected in the standard derivation of the Van Cittert–Zernike theorem, but on the other hand one may hope that it could be employed for a temporal aperture synthesis, where virtual baselines are created through the motion of the satellite. Here, we generalize the formulation of the aperture synthesis concept to the case of observers moving with respect to the sources, and to the correlation of fields measured at times that differ by the travel time of the observer along a virtual baseline. Our derivation is based on first principles, starting with the wave propagation in the Earth reference frame of electro-magnetic fields arising from incoherent current sources, and Lorentz transforming the fields into the reference frame of the satellite. Our detailed study leads to the remarkable conclusion that the delay time due to observer motion cancels exactly the Doppler effect. This justifies the neglect of the Doppler effect in existing imaging systems based on the standard Van Cittert–Zernike theorem.

1. Introduction

Passive microwave remote sensing has a long track record especially in Earth observation. Currently, there is an increasing interest in using microwave remote sensing for monitoring key parameters of our environment. It offers an all-weather capability that is paramount to understanding and monitoring surface variables and parameters of Earth. Among the parameters which can be measured are sea surface temperature [1], ocean salinity [2], soil moisture [3] and sea ice concentration [4, 5] to name but a few. For several measurements, notably sea surface salinity or surface soil moisture, low frequencies are to be preferred [6, 7] which leads to using the protected frequency bands (e.g., 1400-1427 MHz). However, spatial resolution then becomes an issue because an adequate resolution (10 – 40 km) requires a large antenna (respectively 32 to 8 meters, roughly). As the launcher’s shroud is limited, embarking an 8 meter antenna is already

a challenge which has been addressed differently according to the science objectives.

The SMOS mission was the first attempt ever to have an L Band radiometer on a satellite (the SkyLab mission was an attempt to check feasibility during a very short period of time in the 1970s [8]). To address both sea surface salinity and soil moisture it was proposed to ESA to use spatial aperture synthesis interferometry [9], giving a relatively high spatial resolution (ranging from 27 to 60 km) with a deployable antenna. The radiometric resolution is relatively low but can easily be compensated by multi-angular acquisitions [21]. The satellite was launched in 2009 and is still operational providing good results on both oceans and land. Alternative approaches use filled antennae and do not achieve the same spatial resolution, such as the Aquarius mission (launched 2011, ~ 100 km resolution, [18]), and the SMAP mission (launched in January, 2015, 47×61 km resolution [19]).

Spatial aperture synthesis is a widespread technique in radio-astronomy [20]. It is based on precisely timing the arrival times of signals from a given source at spatially separated antennae and then correlating them. In the simplest case of negligible motion of the observer relative to the source and correlation of simultaneously registered signals the correlation functions are related by a simply Fourier-type law to the intensity distribution of the distant incoherent source through the Van Cittert–Zernike theorem (VCZT) (see eq.(29) below).

The standard derivations of the theorem do not take the motion of the observer relative to the sources into account. In particular, the Doppler shift is neglected. Secondly, one may want to generalize the theorem to correlations of electric fields measured at *different* times in order to achieve a *temporal aperture synthesis*, where in addition to the physical baselines provided by different antennae, virtual baselines are created through the motion of the same antenna. In astronomy, rotational synthesis using the motion of Earth has long been used, starting from radio-astronomy in the 1950s till its advance into the visible in the last decade [17, 12]. Satellite-based temporal aperture synthesis for radio astronomy is currently being tested on board of the mission Spektr-R (Radio-Astron) [10], whose measured signals are combined with those of antennae on Earth and allow one to create virtual baselines of up to 350,000 km lengths. The notion of measurements at different times arises here due to the different arrival times of a signal from a given small source. Also in microwave remote sensing of Earth the use of motion of the satellite was proposed in the recent past [11, 14] with the same idea of delays specific for each antenna and pixel.

The purpose of this paper is two-fold: Firstly, we provide the appropriate generalizations of the Van Cittert–Zernike theorem to the case of an observer moving relative to the sources. Secondly, we investigate correlations of fields measured at different times, delayed by the motion of the observer along the desired virtual base line. These delay times depend on the motion of the antennae, but not on the positions of the sources. The question is whether with such global, pixel-independent shifts virtual

baselines can be created. Our derivation is based on first principles, using only Maxwell’s equations and a Lorentz transformation from the Earth frame (where the sources of the electro-magnetic fields are at rest) to the satellite frame, taken to be moving uniformly with constant speed relative to Earth. We show that there is a remarkable cancellation of the phase due to the first order Doppler effect and the phase due to the virtual baseline. This renders the idea of temporal aperture synthesis through direct correlation of electric fields shifted in time by the travel time of the satellite over the distance of the virtual baseline impossible. At the same time, our generalized derivation shows that the VCZT in its standard form is still valid to first order in the speed of the observer and provides therefore a *a posteriori* justification of the use of the theorem for satellite observations of Earth without including the Doppler shift.

2. Doppler effect

Consider two antennae on-board a satellite flying at a height h with a speed v_s in the along-track (x -direction) over a current sources at Earth surface. Let $\mathcal{R} = (O, \hat{e}_1, \hat{e}_2, \hat{e}_3)$ be the reference frame fixed with respect to Earth and $\mathcal{R}' = (O', \hat{e}'_1, \hat{e}'_2, \hat{e}'_3)$ the reference frame moving with speed \mathbf{v}_s , in which the satellite is thus at rest with the antennae at fixed positions \mathbf{r}' . At time $t = t' = 0$, the two origins O, O' are taken to coincide. Let $\mathbf{j}(\mathbf{r}, t)$ be a current density expressed at the position $\mathbf{r} = (x_1, x_2, x_3)$ in the reference frame \mathcal{R} relative to Earth. One is interested in calculating electric fields at points $\mathbf{r}' = (x'_1, x'_2, x'_3)$ relative to \mathcal{R}' , and with time stamps t' , the time measured by a clock on board of the satellite. Two types of baselines are created when considering two antennae at fixed positions $\mathbf{r}'_1, \mathbf{r}'_2$ with respect to \mathcal{R}' : A real physical baseline between the two antennae and a virtual baseline due to the displacement of the antenna. The first one, exploited in spatial aperture synthesis and implemented in SMOS, uses fields measured at space-time points (\mathbf{r}'_1, t') and (\mathbf{r}'_2, t') , implying a physical baseline $\mathbf{r}'_1 - \mathbf{r}'_2$. The second one uses measurements recorded in space-time points (\mathbf{r}'_1, t'_1) and (\mathbf{r}'_2, t'_2) for two different times t'_1, t'_2 , which implies a virtual baseline in frame \mathcal{R} with an additional “virtual” component $\simeq v_s(t'_2 - t'_1)$ in the direction of the motion of the satellite (neglecting here for clarity the relativistic correction, see below for the precise transformation). This is therefore a “temporal” aperture synthesis.

We now calculate the electric fields measured by the satellite that are created by charge and current densities in the Earth-fixed frame \mathcal{R} .

2.1. Lorentz transformation

The sources of the e.m. fields are microscopic charges and currents at the surface of Earth’s surface, in thermal equilibrium at some position dependent temperature T . They give rise to an electric field $\mathbf{E}(\mathbf{r}, t)$, and a magnetic induction $\mathbf{B}(\mathbf{r}, t)$, originally expressed in the frame \mathcal{R} fixed to Earth. These fields are observed at a space-time point (\mathbf{r}', t') in the frame \mathcal{R}' relative to the satellite. The fields $\mathbf{E}'(\mathbf{r}', t')$ measured in the

satellite frame \mathcal{R}' can be obtained from a Lorentz transformation (LT) that describes the relation between physical quantities measured by two observers moving at constant speed with respect to each other [13]. Electric and magnetic fields mix under the Lorentz transformation, i.e. we need to calculate both $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ for obtaining $\mathbf{E}'(\mathbf{r}', t')$. More precisely, the strategy will be the following:

- Calculate $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ generated by $\mathbf{j}(\mathbf{r}, t)$ for every space-time point in \mathcal{R} .
- Apply a Lorentz transformation of these two fields in order to get $\mathbf{E}'(\mathbf{r}, t)$, the electric field in the frame \mathcal{R}' for every space-time point in \mathcal{R} .
- Calculate the four-vector (ct', \mathbf{r}') in the moving frame by Lorentz transforming (ct, \mathbf{r}) in order to have $\mathbf{E}'(\mathbf{r}', t') = \mathbf{E}'(LT(\mathbf{r}), LT(t))$ in \mathcal{R}' .

We neglect the general relativistic effects due to the acceleration on the elliptic orbit of the satellite and assume that the satellite flies with constant speed on a straight line over a plane.

For a given time dependent charge density $\rho(\mathbf{r}, t)$ and current density $\mathbf{j}(\mathbf{r}, t)$, the exact expressions for $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ are (see eqs.(6.55,6.56) in 5th German edition of [13])

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int d^3r'' \left\{ \frac{\hat{R}}{R^2} \rho(\mathbf{r}'', t - R/c) + \frac{\hat{R}}{cR} \frac{\partial \rho(\mathbf{r}'', t - R/c)}{\partial t} - \frac{1}{c^2 R} \frac{\mathbf{j}(\mathbf{r}'', t - \frac{R}{c})}{\partial t} \right\}, \quad (1)$$

$$\mathbf{B}(\mathbf{r}, t) = -\frac{\mu_0}{4\pi} \int d^3r'' \frac{\hat{R}}{R} \times \left(\frac{1}{R} \mathbf{j}(\mathbf{r}'', t - \frac{R}{c}) + \frac{1}{c} \frac{\partial \mathbf{j}(\mathbf{r}'', t - \frac{R}{c})}{\partial t} \right), \quad (2)$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}''$, $R = |\mathbf{R}|$, \hat{R}/R is the unit vector in $(\mathbf{r} - \mathbf{r}'')$ -direction, and ϵ_0, μ_0 are the electric susceptibility and magnetic permeability of vacuum, respectively. Eqs.(1,2) show that the electric and magnetic fields are fully retarded with respect to the sources by the propagation time R/c . Both fields contain terms present also for static charges and currents that decay according to the Coulomb-law with distance, i.e. as $1/R^2$. However, in the far field $R \gg \lambda$, where λ is the wavelength, these terms are negligible compared to the ones decaying as $1/R$ and which arise only for time-dependent charge and current densities [22]. In the following we will take into account only these radiation fields. In the far-field approximation, the fields are given by

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int d^3r'' \left\{ \frac{\hat{R}}{cR} \frac{\partial \rho(\mathbf{r}'', t - R/c)}{\partial t} - \frac{1}{c^2 R} \frac{\mathbf{j}(\mathbf{r}'', t - \frac{R}{c})}{\partial t} \right\}, \quad (3)$$

$$\mathbf{B}(\mathbf{r}, t) = -\frac{\mu_0}{4\pi} \int d^3r'' \frac{\hat{R}}{R} \times \frac{1}{c} \partial_t \mathbf{j}(\mathbf{r}'', t - \frac{R}{c}), \quad (4)$$

The Lorentz transformation of the electric and magnetic fields from \mathcal{R} to \mathcal{R}' reads [13],

$$\mathbf{E}' = \gamma(\mathbf{E} + c\boldsymbol{\beta} \times \mathbf{B}) - \frac{\gamma^2}{\gamma + 1} \boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{E}), \quad (5)$$

$$\mathbf{B}' = \gamma(\mathbf{B} - \boldsymbol{\beta} \times \mathbf{E}/c) - \frac{\gamma^2}{\gamma + 1} \boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{B}), \quad (6)$$

where $\boldsymbol{\beta} = \mathbf{v}_s/c$, $\beta = |\boldsymbol{\beta}|$ and $\gamma = 1/\sqrt{1 - (v_s/c)^2}$. We deduce the expression of the electric field in \mathcal{R}' as function of space-time coordinates in \mathcal{R} ,

$$\begin{aligned} \mathbf{E}'(\mathbf{r}, t) = & -\frac{\gamma\mu_0}{4\pi} \int \frac{d^3\mathbf{r}''}{R} \left[\partial_t(\mathbf{j} - c\hat{R}\rho) + (\boldsymbol{\beta} \cdot \partial_t\mathbf{j})\hat{R} - (\boldsymbol{\beta} \cdot \hat{R})\partial_t\mathbf{j} \right. \\ & \left. - \frac{\gamma}{\gamma + 1} \boldsymbol{\beta} \cdot \partial_t(\mathbf{j} - c\hat{R}\rho)\boldsymbol{\beta} \right], \end{aligned} \quad (7)$$

where the arguments of ρ and \mathbf{j}^t are $(\mathbf{r}'', t - \frac{R}{c})$.

The next step consists of Lorentz transforming the space-time coordinates in \mathcal{R} in order to get the expressions of (\mathbf{r}, t) as function of (\mathbf{r}', t') ,

$$t = \gamma(t' + \frac{\boldsymbol{\beta} \cdot \mathbf{r}'}{c}), \quad (8)$$

$$\mathbf{r} = \mathbf{r}' + \frac{\gamma - 1}{\beta^2}(\mathbf{r}' \cdot \boldsymbol{\beta})\boldsymbol{\beta} + \gamma\boldsymbol{\beta}ct'. \quad (9)$$

Inserting (8,9) in (7), one finds that $\mathbf{E}'(\mathbf{r}', t')$ is still given by (7), where, however, the arguments of ρ and \mathbf{j}^t , to be inserted after differentiation with respect to t , are now $(\mathbf{r}'', \gamma(t' + \frac{\boldsymbol{\beta} \cdot \mathbf{r}'}{c}) - \frac{R}{c})$, and \mathbf{r} is to be replaced everywhere on the right hand side, including in \mathbf{R} , R , and \hat{R} , by expression (9).

2.2. Expansion in β

Eq. (7) can still be further simplified if we take into account that β is typically very small. The example of SMOS with $h \simeq 700$ km, $v_s \simeq 7$ km/s, gives $\beta \sim v_s/c \sim 10^{-5}$. It makes therefore sense to expand in powers of β . We keep in the following systematically terms up to order β in all phases, but neglect corrections of order β in the prefactor. This is justified by the fact that in the end we calculate correlation functions, where the phase information is crucial, whereas the overall amplitudes are irrelevant, as they are, in practice, always renormalized with respect to the total intensity.

To first order in β , we have $\gamma \simeq 1$ and the following approximations

$$t \simeq t' + \frac{\boldsymbol{\beta} \cdot \mathbf{r}'}{c} \quad (10)$$

$$\mathbf{r} \simeq \mathbf{r}' + \mathbf{v}_st' \Rightarrow R \simeq |\mathbf{r}' + \mathbf{v}_st' - \mathbf{r}''| \quad (11)$$

With this, one finds

$$\mathbf{E}'(\mathbf{r}', t') \simeq -\frac{\mu_0}{4\pi} \int \frac{d^3\mathbf{r}''}{R} \partial_t \left[\mathbf{J}(\mathbf{r}'', t) \right]_{t=t' + \frac{\boldsymbol{\beta} \cdot \mathbf{r}' - R}{c}}, \quad (12)$$

where we have introduced a total current source $\mathbf{J}(\mathbf{r}'', t) = \mathbf{j}(\mathbf{r}'', t) - \hat{R}c\rho(\mathbf{r}'', t)$ and $R = |\mathbf{r}' - \mathbf{r}'' + \mathbf{v}_st'|$. If the microscopic charge density vanishes, $\rho = 0$, then the continuity equation $\text{div}\mathbf{j} + \partial_t\rho = 0$ implies $\text{div}\mathbf{j} = 0$. In this case, the current density is purely transverse, $\mathbf{J} = \mathbf{j} = \mathbf{j}^t$, where \mathbf{j}^t denotes the part with vanishing divergence. In

general, however, fluctuating microscopic charges can temporarily accumulate, in which case the part $\partial_t \hat{R} c \rho$ may rapidly dominate over $\partial_t \mathbf{j}$, as $\mathbf{j} = \rho \mathbf{v}$ for charges moving with a speed \mathbf{v} , and typically $v \ll c$. For the further derivation, there is no need to distinguish the two cases, and we therefore keep working with the total current density \mathbf{J} .

Another approximation is in order: We will be interested only in times t' smaller or equal to the maximum possible averaging time. As explained below, this time is set by the time of overflight of a single pixel, assumed to be at constant temperature. With a pixel dimension $\Delta x'' \sim \Delta y'' \sim$ some 1-10km, we have $v_s t' / |\mathbf{r}' - \mathbf{r}''| \leq \Delta x'' / h \ll 1$. For the typical values of SMOS, the maximum t' would be of order 1s, but the inequality shows that even substantially longer times ($t \lesssim 100$ s) can still be accommodated in the range of validity of the following approximation:

$$\begin{aligned}
 R &\simeq \sqrt{(\mathbf{r}' + \mathbf{v}_s t' - \mathbf{r}'')^2} \\
 &= |\mathbf{r}' - \mathbf{r}''| \sqrt{1 + 2 \frac{(\mathbf{r}' - \mathbf{r}'') \cdot \mathbf{v}_s t'}{|\mathbf{r}' - \mathbf{r}''|^2} + \left(\frac{\mathbf{v}_s t'}{|\mathbf{r}' - \mathbf{r}''|} \right)^2} \\
 &\simeq |\mathbf{r}' - \mathbf{r}''| \left(1 + \frac{(\mathbf{r}' - \mathbf{r}'') \cdot \mathbf{v}_s t'}{|\mathbf{r}' - \mathbf{r}''|^2} \right) \\
 &= |\mathbf{r}' - \mathbf{r}''| + \hat{e}_{\mathbf{r}' - \mathbf{r}''} \cdot \mathbf{v}_s t'.
 \end{aligned} \tag{13}$$

As for the shift $\boldsymbol{\beta} \cdot \mathbf{r}' / c$ in the time argument, we have, with antennae at positions (x'_i, y'_i, h) and $|x'_i|, |y'_i| \simeq (10-100)\text{m} \ll v_s t'$, $|\boldsymbol{\beta} \cdot \mathbf{r}' / c| \ll \beta^2 t'$ for almost all t' , such that this term can also be neglected. Also note that this term vanishes exactly for antennae located on a line perpendicular to the displacement of the satellite, $\boldsymbol{\beta} \cdot \mathbf{r}' = 0$.

By neglecting again corrections of the amplitude of order β or higher, the expression of the received electric field at position \mathbf{r}' of the antenna relative to the satellite, expressed in the frame \mathcal{R}' , becomes

$$\mathbf{E}'(\mathbf{r}', t') \simeq -\frac{\mu_0}{4\pi} \int \frac{d^3 \mathbf{r}''}{|\mathbf{r}' - \mathbf{r}''|} \partial_t \mathbf{J}(\mathbf{r}'', t) \Big|_{t=t' - \frac{R(t')}{c}}, \tag{14}$$

with $R(t') = |\mathbf{r}' + \mathbf{v}_s t' - \mathbf{r}''|$. Analyzing expression (14), one realizes that it could have been obtained in a naive way by simply replacing \mathbf{r} in the far field expression of $\mathbf{E}(\mathbf{r}, t)$ with the position of the moving satellite as seen in \mathcal{R} , and keeping the time t . However, even the meaning of the partial time-derivative would have then remained ambiguous: does it apply to $R(t)$ or not? In the formal, rigorous derivation that we have followed here, it is clear, that it does not: one first differentiates $\mathbf{J}(\mathbf{r}'', t)$ with respect to t and then replaces t as indicated. Also, the sequence of well-controlled approximations discussed above opens the way to systematically deriving higher-order corrections, which is clearly not possible in the mentioned “naive” approach.

2.3. Fourier transform

Thermal sources at the surface of Earth, which are the origin of the thermal noise detected at the position of the satellite, are assimilated to random fluctuations as

function of time. They can be expressed through their spectrum $\tilde{\mathbf{J}}$,

$$\mathbf{J}(\mathbf{r}'', t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega' e^{i\omega' t} \tilde{\mathbf{J}}(\mathbf{r}'', \omega'). \quad (15)$$

Inserting (15) in (14) yields

$$\mathbf{E}'(\mathbf{r}', t') \simeq -\frac{\mu_0}{4\pi\sqrt{2\pi}} \int \frac{d^3 r''}{|\mathbf{r}' - \mathbf{r}''|} \int d\omega' i\omega' e^{i\omega'(t' - \frac{R(t')}{c})} \tilde{\mathbf{J}}(\mathbf{r}'', \omega'). \quad (16)$$

The spectrum of the electric field received at the position of the satellite corresponds to the Fourier transform of (16),

$$\tilde{\mathbf{E}}'(\mathbf{r}', \omega) \simeq -\frac{\mu_0}{8\pi^2} \int dt' e^{-i\omega t'} \int \frac{d^3 r''}{|\mathbf{r}' - \mathbf{r}''|} \int d\omega' i\omega' e^{i\omega'(t' - \frac{R(t')}{c})} \tilde{\mathbf{J}}(\mathbf{r}'', \omega'), \quad (17)$$

and using the approximation (13), the previous expression becomes

$$\begin{aligned} \tilde{\mathbf{E}}'(\mathbf{r}', \omega) \simeq & -\frac{\mu_0}{8\pi^2} \int \frac{d^3 r''}{|\mathbf{r}' - \mathbf{r}''|} \int d\omega' i\omega' e^{-i\omega' \frac{|\mathbf{r}' - \mathbf{r}''|}{c}} \tilde{\mathbf{J}}(\mathbf{r}'', \omega') \\ & \times \int dt' e^{-i\omega t'} e^{i\omega'(1 - \hat{\mathbf{e}}_{\mathbf{r}' - \mathbf{r}''} \cdot \boldsymbol{\beta}) t'}. \end{aligned} \quad (18)$$

By noticing that the only remaining time dependence is in the exponent, the integral over time variable t' can thus be simplified to

$$\begin{aligned} \int dt' e^{-i\omega t'} e^{i\omega'(1 - \hat{\mathbf{e}}_{\mathbf{r}' - \mathbf{r}''} \cdot \boldsymbol{\beta}) t'} &= 2\pi \delta(-\omega + \omega'(1 - \hat{\mathbf{e}}_{\mathbf{r}' - \mathbf{r}''} \cdot \boldsymbol{\beta})) \\ &= \frac{2\pi}{|1 - \hat{\mathbf{e}}_{\mathbf{r}' - \mathbf{r}''} \cdot \boldsymbol{\beta}|} \delta(\omega' - \omega(1 - \hat{\mathbf{e}}_{\mathbf{r}' - \mathbf{r}''} \cdot \boldsymbol{\beta})^{-1}) \\ &\simeq 2\pi |1 + \hat{\mathbf{e}}_{\mathbf{r}' - \mathbf{r}''} \cdot \boldsymbol{\beta}| \delta(\omega' - \omega(1 + \hat{\mathbf{e}}_{\mathbf{r}' - \mathbf{r}''} \cdot \boldsymbol{\beta})), \end{aligned}$$

where the last expression is once more correct to order β . By neglecting again corrections of the electric field amplitude, the spectrum of the electric field measured in \mathcal{R}' becomes

$$\tilde{\mathbf{E}}'(\mathbf{r}', \omega) \simeq -\frac{\mu_0}{4\pi} \int d^3 r'' \frac{i\omega e^{-i\omega \frac{|\mathbf{r}' - \mathbf{r}''|}{c}} (1 + \hat{\mathbf{e}}_{\mathbf{r}' - \mathbf{r}''} \cdot \boldsymbol{\beta})}{|\mathbf{r}' - \mathbf{r}''|} \tilde{\mathbf{J}}(\mathbf{r}'', \omega(1 + \hat{\mathbf{e}}_{\mathbf{r}' - \mathbf{r}''} \cdot \boldsymbol{\beta})).$$

On board of the satellite, the received electric field typically passes through a filter with filter function $w(\omega)$. E.g. for SMOS, the spectrum is filtered to a narrow window corresponding to the allowed band of width $b = 2\pi \times 17\text{MHz}$ centered at $\omega_0 = 2\pi \times 1.4135\text{GHz}$. The retrieved temporal electric field at the output of the filter is expressed through inverse Fourier transforming $\tilde{\mathbf{E}}'(\mathbf{r}', \omega)$ multiplied with the filter function,

$$\begin{aligned} \mathbf{E}'(\mathbf{r}', t') &= \frac{1}{\sqrt{2\pi}} \int d\omega w(\omega) \tilde{\mathbf{E}}'(\mathbf{r}', \omega) e^{i\omega t'} \\ &\simeq -\frac{\mu_0}{4\pi} \frac{1}{\sqrt{2\pi}} \int d\omega w(\omega) \int d^3 r'' \frac{i\omega}{|\mathbf{r}' - \mathbf{r}''|} e^{i\omega t'} \\ &\quad \times \tilde{\mathbf{J}}(\mathbf{r}'', \omega(1 + \hat{\mathbf{e}}_{\mathbf{r}' - \mathbf{r}''} \cdot \boldsymbol{\beta})) e^{-i\omega \frac{|\mathbf{r}' - \mathbf{r}''|}{c}} (1 + \hat{\mathbf{e}}_{\mathbf{r}' - \mathbf{r}''} \cdot \boldsymbol{\beta}). \end{aligned} \quad (19)$$

Since the integration over the frequency variable ω is from $-\infty$ to $+\infty$, one can apply the change of variables $\omega(1 + \hat{\mathbf{e}}_{\mathbf{r}' - \mathbf{r}''} \cdot \boldsymbol{\beta}) \rightarrow \omega$. Neglecting once more corrections of the

amplitude of order β or higher, and corrections of the phase of order β^2 or higher, this leads to

$$\begin{aligned} \mathbf{E}'(\mathbf{r}', t') \simeq & -\frac{\mu_0}{4\pi} \frac{1}{\sqrt{2\pi}} \int i\omega d\omega \int d^3r'' \tilde{\mathbf{J}}(\mathbf{r}'', \omega) \\ & \times \frac{w(\omega(1 - \hat{\mathbf{e}}_{\mathbf{r}' - \mathbf{r}''} \cdot \boldsymbol{\beta}))}{|\mathbf{r}' - \mathbf{r}''|} e^{i\omega(t' - \frac{|\mathbf{r}' - \mathbf{r}''|}{c})} e^{-i\omega t' \hat{\mathbf{e}}_{\mathbf{r}' - \mathbf{r}''} \cdot \boldsymbol{\beta}}. \end{aligned} \quad (20)$$

As it was to be expected, the satellite motion generates a Doppler shift of the frequency of the sources. The Doppler shift appears in the expression of the filtered electric field by means of a shift of the frequency of the filter function and the appearance of the phase factor $e^{-i\omega t' \hat{\mathbf{e}}_{\mathbf{r}' - \mathbf{r}''} \cdot \boldsymbol{\beta}}$. The Doppler shift to linear order in β is clearly a purely longitudinal one, as is well-known. A transverse (i.e. purely relativistic) Doppler shift would appear at second order in β .

2.4. Correlation function

We define the correlation (also called the visibility function) of the electric fields in the frame \mathcal{R}' fixed to the satellite as

$$C(\mathbf{r}'_1, t'_1, \mathbf{r}'_2, t'_2) \equiv \langle \mathbf{E}'(\mathbf{r}'_1, t'_1) \mathbf{E}'^*(\mathbf{r}'_2, t'_2) \rangle. \quad (21)$$

The thermal sources at the surface of Earth can be modeled by Gaussian stochastic processes that are uncorrelated for different frequencies and positions [15, 16],

$$\langle \tilde{\mathbf{J}}(\mathbf{r}''_1, \omega_1) \tilde{\mathbf{J}}^*(\mathbf{r}''_2, \omega_2) \rangle = \frac{l_c^3}{T_c} \delta(\mathbf{r}''_1 - \mathbf{r}''_2) \delta(\omega_1 - \omega_2) \langle |\tilde{\mathbf{J}}(\mathbf{r}''_1, \omega_1)|^2 \rangle, \quad (22)$$

where l_c and T_c refer to the correlation length and the correlation time, respectively. The averages in (21) and (22) are in principle over an ensemble of different realizations of the noise processes, but, assuming ergodicity, they may be replaced by a time average. The averaging time should be as long as possible to reduce the fluctuations of the average, but sufficiently short for not mixing different inequivalent ensembles. In our case this means that the averaging time should be comparable to the time it takes for the satellite to fly over one pixel (with assumed constant temperature). This renders the definition (21) operational for a single pass of the satellite.

Let $\Delta\mathbf{r} = \mathbf{r}'_2 - \mathbf{r}'_1$ and $\Delta t = t'_2 - t'_1$. To first order in $|\Delta\mathbf{r}|/|\mathbf{r}'_1 - \mathbf{r}''| \sim 10^{-4}$, we have the following approximations

$$|\mathbf{r}'_2 - \mathbf{r}''| \simeq |\mathbf{r}'_1 - \mathbf{r}''| + \Delta\mathbf{r} \cdot \hat{\mathbf{e}}_{\mathbf{r}'_1 - \mathbf{r}''}, \quad (23)$$

$$\hat{\mathbf{e}}_{\mathbf{r}'_2 - \mathbf{r}''} \cdot \boldsymbol{\beta} \simeq \hat{\mathbf{e}}_{\mathbf{r}'_1 - \mathbf{r}''} \cdot \boldsymbol{\beta}. \quad (24)$$

Finally, using (18), (22), (23) and (24), the expression (21) of the correlation function becomes

$$\begin{aligned} C(\mathbf{r}'_1, t'_1, \mathbf{r}'_2, t'_2) \simeq & \frac{l_c^3}{2\pi T_c} \left(\frac{\mu_0}{4\pi}\right)^2 \int d\omega \omega^2 \int d^3r'' \langle |\tilde{\mathbf{J}}(\mathbf{r}'', \omega)|^2 \rangle \\ & \times \frac{|w(\omega(1 - \hat{\mathbf{e}}_{\mathbf{r}'_1 - \mathbf{r}''} \cdot \boldsymbol{\beta}))|^2}{|\mathbf{r}'_1 - \mathbf{r}''| |\mathbf{r}'_2 - \mathbf{r}''|} \exp[-i\omega\Delta t + i\frac{\omega}{c}(\Delta\mathbf{r} + \Delta t \mathbf{v}_s) \cdot \hat{\mathbf{e}}_{\mathbf{r}'_1 - \mathbf{r}''}]. \end{aligned} \quad (25)$$

We assume that $\omega^2 \langle |\tilde{\mathbf{J}}(\mathbf{r}'', \omega)|^2 \rangle$ which is related to the brilliance temperature depends only weakly on frequency (compared to the rapid oscillations of the phase as function of ω) over the bandwidth b , $\omega^2 \langle |\tilde{\mathbf{J}}(\mathbf{r}'', \omega)|^2 \rangle \simeq \omega_0^2 \langle |\tilde{\mathbf{J}}(\mathbf{r}'', \omega_0)|^2 \rangle$. It is then convenient to invert the applied change of variables $\omega(1 - \hat{\mathbf{e}}_{\mathbf{r}'_1 - \mathbf{r}''} \cdot \boldsymbol{\beta}) \rightarrow \omega$, and one easily finds

$$C(\mathbf{r}'_1, t'_1, \mathbf{r}'_2, t'_2) \simeq \frac{l_c^3}{2\pi T_c} \left(\frac{\mu_0}{4\pi}\right)^2 \int d\omega \omega_0^2 |w(\omega)|^2 \int d^3 r'' \frac{\langle |\tilde{\mathbf{J}}(\mathbf{r}'', \omega_0)|^2 \rangle}{|\mathbf{r}'_1 - \mathbf{r}''| |\mathbf{r}'_2 - \mathbf{r}''|} \quad (26)$$

$$\times \exp[-i\omega \Delta t - i\omega \Delta t \hat{\mathbf{e}}_{\mathbf{r}'_1 - \mathbf{r}''} \cdot \boldsymbol{\beta} + i\frac{\omega}{c}(\Delta \mathbf{r} + \Delta t \mathbf{v}_s) \cdot \hat{\mathbf{e}}_{\mathbf{r}'_1 - \mathbf{r}''}].$$

By noticing that $\boldsymbol{\beta} = \mathbf{v}_s/c$ and neglecting once more corrections of order β^2 in the phase, it follows that

$$C(\mathbf{r}'_1, t'_1, \mathbf{r}'_2, t'_2) \simeq \frac{l_c^3}{2\pi T_c} \left(\frac{\mu_0}{4\pi}\right)^2 \int d\omega \omega_0^2 |w(\omega)|^2 \quad (27)$$

$$\times \int d^3 r'' \frac{\langle |\tilde{\mathbf{J}}(\mathbf{r}'', \omega_0)|^2 \rangle}{|\mathbf{r}'_1 - \mathbf{r}''| |\mathbf{r}'_2 - \mathbf{r}''|} \exp[-i\omega \Delta t + i\frac{\omega}{c} \Delta \mathbf{r} \cdot \hat{\mathbf{e}}_{\mathbf{r}'_1 - \mathbf{r}''}].$$

We clearly see that the two phases $-\omega \Delta t \hat{\mathbf{e}}_{\mathbf{r}'_1 - \mathbf{r}''} \cdot \boldsymbol{\beta}$ of the Doppler shift and $(\omega/c) \Delta t \mathbf{v}_s \cdot \hat{\mathbf{e}}_{\mathbf{r}'_1 - \mathbf{r}''}$ corresponding to the virtual baseline in the direction of the motion of the satellite cancel.

Finally, by considering a simple rectangular filter function of bandwidth b ,

$$w(\omega) = \begin{cases} 1 & \text{for } \omega_0 - b/2 \leq \omega \leq \omega_0 + b/2, \\ 0 & \text{elsewhere,} \end{cases}$$

the integral over ω can be performed. To first order in β , one finds

$$C(\mathbf{r}'_1, t'_1, \mathbf{r}'_2, t'_2) \simeq K \int d^3 r'' \langle |\tilde{\mathbf{J}}(\mathbf{r}'', \omega_0)|^2 \rangle$$

$$\times \frac{\exp[i\omega_0(-\Delta t + \frac{1}{c} \Delta \mathbf{r} \cdot \hat{\mathbf{e}}_{\mathbf{r}'_1 - \mathbf{r}''})]}{|\mathbf{r}'_1 - \mathbf{r}''| |\mathbf{r}'_2 - \mathbf{r}''|}$$

$$\times \text{sinc}[\frac{b}{2}(-\Delta t + \frac{1}{c} \Delta \mathbf{r} \cdot \hat{\mathbf{e}}_{\mathbf{r}'_1 - \mathbf{r}''})]. \quad (28)$$

where $\text{sinc}(x) \equiv \sin(x)/x$, and the constant $K = l_c^3 b \omega_0^2 \mu_0^2 / (32\pi^3 T_c)$. Eq.(28) is our main result. It generalizes the Van Cittert–Zernike theorem to an observer moving with respect to the sources and to a finite time-interval Δt between the measurements of the electric fields, as we discuss now.

3. Discussion

A passive micro-wave interferometer for Earth observation measures the complex spatial correlation field, or the visibility function, of the incident electric field originating from thermally fluctuating sources on Earth's surface. The Van Cittert–Zernike theorem describes the Fourier transform relationship between a spatial intensity distribution

of these incoherent sources of radiation and its associated visibility function. In our notation, the theorem can be written as

$$C_{VCZ}(\mathbf{r}'_1, t'_1, \mathbf{r}'_2, t'_2) \simeq K \int d^3r'' \langle |\tilde{\mathbf{J}}(\mathbf{r}'', \omega_0)|^2 \rangle \frac{\exp [i\omega_0(\frac{1}{c}\Delta\mathbf{r} \cdot \hat{\mathbf{e}}_{\mathbf{r}'_1 - \mathbf{r}''})]}{|\mathbf{r}'_1 - \mathbf{r}''||\mathbf{r}'_2 - \mathbf{r}''|}. \quad (29)$$

It shows that the set of equal time visibility functions obtained from different antennae pairs is given by the spatial 2D Fourier transform of the intensity distribution of the sources, where the phases of the Fourier transform are the scalar products of the wave-vector from the source to an antenna and the vector joining two antennae.

In the standard derivation of the theorem, sources and observer are taken at rest with respect to each other, and only electric fields observed at the same time (with respect to the observer reference-frame) but at different positions are correlated. We recover the standard form of the VCZT when setting $\Delta t = 0$, and considering a small bandwidth, $b\Delta\mathbf{r}/c \ll 1$. The latter condition allows one to approximate the sinc-function by $\text{sinc}(x) \simeq 1$. The speed of the satellite v_s has disappeared from the expression already in (27) with the cancellation of the Doppler shift and the phase related to the virtual baseline created by the displacement of the satellite during time Δt . Thus, *the standard form (29) is valid also at finite speed to first order in β* . Our study therefore extends the validity of the standard form (29) of the Van Cittert–Zernike theorem to the case of an observer moving with constant speed with respect to the sources.

The cancellation of the phases due to the Doppler shift and the virtual baseline shows that it is not possible to create a temporal aperture synthesis by correlating the observed time-dependent electric fields delayed by the travel time of the satellite in the direction of the virtual baseline. In addition, a finite time interval Δt between two observations leads to *i.)* a strong suppression of the amplitude of the correlation function and *ii.)* a rapidly oscillating phase factor. The first effect results from the sinc-term. If $\Delta t \sim \Delta\mathbf{r}/v_s$, the term due to Δt in the argument of the sinc is up to a factor c/v_s larger than the second one. The second one, on the other hand, has to be of order one if the spatial aperture in the direction of the real baseline $\Delta\mathbf{r}$ is supposed to work. Therefore, the amplitude of the correlation function is suppressed by a factor $\sim v_s/c$ relative to the standard case with $\Delta t = 0$. The rapidly oscillating phase factor is given by $\exp(-i\omega_0\Delta t)$. This phase overwhelms the information in the cross-track direction contained in the phase $(\omega_0/c)\Delta\mathbf{r} \cdot \hat{\mathbf{e}}_{\mathbf{r}'_1 - \mathbf{r}''}$.

Thus, while the hope of being able to use the virtual baseline created by a moving satellite for imaging purposes through correlating the time-dependent fields shifted only by the travel time over the virtual baseline is disappointed, our derivation justifies the neglect of the Doppler effect in existing satellite-based passive radiometers based on the standard Van Cittert–Zernike theorem [21].

4. Conclusion

We have examined the possibility of temporal aperture synthesis for satellite-based passive microwave observation of Earth, where a virtual baseline is created by the motion of the satellite in order to enhance the spatial resolution. Our study shows that the interesting phase information in the along-track direction obtained from a time shift of the fields corresponding to the travel time over the virtual baseline is exactly canceled by the first order (longitudinal) Doppler effect. Furthermore, the time shift yields a large uncompensated frequency-dependent phase overwhelming the information in the cross-track direction, and a drastic reduction of the amplitude of the correlation function. Therefore, by correlating in this way the time-dependent signals received by a 1D antennae array with pixel-independent shifts corresponding only to the travel time of the satellite, one cannot reconstruct the brilliance temperature in both along- and cross-track directions.

Nevertheless, our result (28) constitutes a generalization of the Van Cittert–Zernike theorem (29) to the case of an observer moving with respect to the sources, and the correlation of electric field measurements at different times. By deriving the electric fields in the moving frame from first principles, we have shown that the longitudinal Doppler effect cancels exactly in the correlation function. The standard Van Cittert–Zernike theorem for equal time correlations therefore holds even for a moving observer with substantially different Doppler shifts in different directions of sight.

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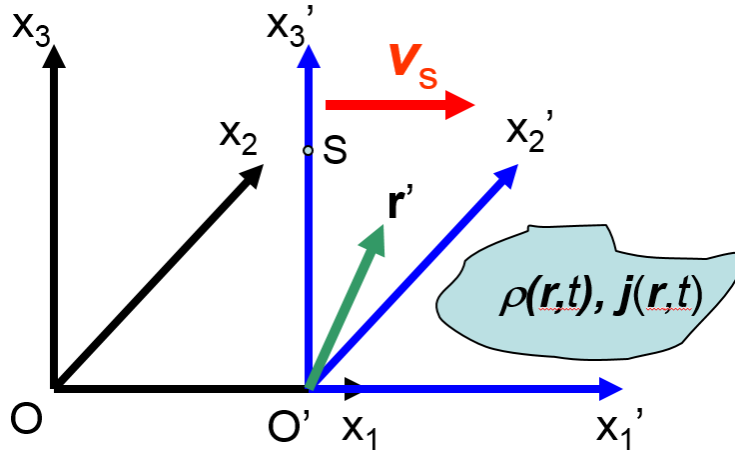


Figure 1. Earth frame $\mathcal{R} = (O, \hat{e}_1, \hat{e}_2, \hat{e}_3)$ and reference frame $\mathcal{R}' = (O', \hat{e}'_1, \hat{e}'_2, \hat{e}'_3)$ moving with speed \mathbf{v}_s relative to \mathcal{R} . Fluctuating charge densities $\rho(\mathbf{r}, t)$ and current densities $\mathbf{j}(\mathbf{r}, t)$ create electric and magnetic fields. Antennae at fixed positions $\mathbf{r}'_i = (x'_i, y'_i, h)$ with respect to \mathcal{R}' measure the electric fields at the proper time t' of the satellite S .